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ADP010895 thru ADP010929

ON THE MULTIFRACTAL PROPERTIES OF PASSIVELY CONVECTED SCALAR FIELDS

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Multifractal spectra are derived for 1- and 2-dimensional cross-sections of passively convected scalar fields; 2- and 3-dimensional single-scale velocity fields in the absence of KAM surfaces are considered. Both the Kraichnan model and real flows with non-zero correlation time are studied. The calculation of $f(\alpha)$-curves is based on the probability density function of the stretching factors of fluid elements. It is shown that strict multifractality holds only for small values of α. New multifractal scalar field — "harmfulness"— is suggested to describe the propagation of environmentally dangerous substances.

Keywords: Turbulent diffusion, passive scalar, intermittence, multifractality.

1 Introduction

The convection of passive scalar by chaotic fluid flow has been studied intensively during the last decades. Special attention has been paid to the analysis of the intermittent structure of scalar fields [1–8]. The scalar density correlation properties have been found to be very far from Gaussian; a clear evidence for this is the multifractal structure of scalar dissipation fields [1–4]

Most of the theoretical studies have been based on the Kraichnan model [5], in which case the velocity field is assumed to be delta correlated in time. Despite of being very idealized, this model is far from trivial and is widely believed to reproduce the most important features of the real passive scalar turbulence. However, this rigorous approach has still been unable to relate the multifractal spectrum $f(\alpha)$ directly to the correlation functions of the velocity field. The main tool for the multifractal analysis has been the generalized Baker map model [which has been also used to calculate the probability density function (PDF) of largest Lyapunov exponents] [2,3]. Recently, new approach has been suggested for the multifractal analysis of the passive scalar dissipation field, which is based on a simple equation describing the stretching of fluid elements [9]. In Sections 2 and 3 we extend the approach to three-dimensional geometry. In Section 4, we define new scalar fields, which are based on the density of the passive scalar, and show that they have also multifractal structure. When the convecting velocity field is oceanic or atmospheric motion, and the passive scalar is an environmentally harmful substance, these fields can be treated as the measures describing the potential damage to the environment.

The parameters of our model equations are defined by the statistical properties of the velocity field. In the case of delta-correlation in time (Kraichnan model), they are directly expressed via the correlation functions. The model is in good agreement with the earlier experimental and theoretical results.

2 Basic equation

The convective diffusion of a passive scalar ϕ is described by the following equation:

$$\frac{\partial \phi}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \phi = \kappa \boldsymbol{\nabla}^2 \phi + g. \tag{1}$$

Here g is a source of passive scalar and κ — the molecular diffusivity. We consider a chaotic isotropic single-scale incompressible velocity field $\boldsymbol{v}(r,t)$ in the absence of KAM surfaces. Single-scale field is defined as a field for which the Fourier spectrum is constrained into one octave of wavelengths. The characteristic space-scale is chosen to be the unit length. We consider large Peclet number limit, $\kappa \ll \langle |\boldsymbol{v}| \rangle$ and moderate time-scale. More precisely, we assume that the convection has created small-scale structures, but the smallest created scales are still longer than the dissipation scale $\sqrt{\kappa t}$, i.e. $t \lesssim -\ln \kappa / 2\gamma$, where γ is the average value of the largest Lyapunov exponent of the velocity field. Then, the stretching of fluid elements and the evolution of passive scalar gradients are related to each other [4,6,10-13]. We study the case when there is no source of dye, and at the initial moment $t = 0$, there was a uniform gradient of dye concentration,

$$g \equiv 0, \qquad \boldsymbol{\nabla} \phi |_{t=0} = \boldsymbol{e}_x, \tag{2}$$

where $\boldsymbol{e}_x$ is the unit vector along x-axis.

With the given assumptions and for 2-dimensional velocity field, the problem of finding the PDF of passive scalar gradients is directly equivalent to the problem of finding the PDF of stretching factors of fluid elements $\rho(r,t) = \exp(h_+ t)\cos\varphi$, where h_+ is the largest Lyapunov exponent and φ — the angle between the respective eigenvector and a fluid element [9]. Indeed, neglecting the seed diffusivity, at a fixed fluid particle, the modulus of the dye gradient $|\boldsymbol{\nabla}\phi|$ scales like the length of an infinitesimal fluid element $\delta r(t)$, initially perpendicular to the gradient $[\delta r(0) \perp \boldsymbol{\nabla}\phi(0)]$: $|\boldsymbol{k}(t)| \propto |\delta r(t)|$. This is due to the incompressibility of the fluid: the fluid parallelogram defined by initially perpendicular vectors $\delta r(t)$ and $\delta r_\perp(t)$ preserves its area $\delta S = |\delta r(t)| \cdot |\delta r_\perp(t)| \sin\alpha$, where α is the angle between the vectors. On the other hand, at the fixed fluid particle, the dye concentration remains unchanged and $|\delta r_\perp(t)| \sin\alpha \propto |\boldsymbol{\nabla}\phi|^{-1}$. Thus, with the proper choice of units and in the absence of seed diffusivity, the stretching factors $\rho = |\delta r(r_0, t)| / |\delta r(r_0, 0)|$ and dye gradients are equivalent to each other. The strict equivalence holds for the fluid elements, initially parallel to the isolines of $\phi(r, 0)$; however, statistically the initial orientation of the fluid elements becomes irrelevant.

Three-dimensional geometry can be treated in a similar way, then the PDF of dye gradients is equivalent to the distribution of stretching factors of fluid surfaces. The argumentation is completely analogous and based on the evolution of fluid parallelepipeds.

The multifractal analysis is based on simple diffusion-convection equation, describing the distribution of fluid elements $l(\rho)$ and fluid surfaces $s(\rho)$ over the stretching factor ρ. We derive it both for the Kraichnan model and for velocity fields of finite correlation time. First we consider the case of real velocity fields of finite correlation time τ. The correlation time is used as the unit time, i.e. $\tau = 1$.

Let us define $l(\rho, t)d\rho$ as the average total length of those pieces of a fluid line, for which $\rho \in [\rho, \rho + d\rho]$ (the length is reduced to the initial length of the fluid line L_0). Then, the PDF of stretching factors and dye gradients is given by $\rho^{-1}l(\rho)$. We consider time increments $\Delta t = 1$, and study the change of the state of fluid elements (fluid surfaces); we neglect the correlation in time for time-scales longer than $\tau = 1$. Let $p(q)dq$ denote the probability of stretching a fluid element by a factor of q. Then we can write $l(\rho, t+1)d\rho = \int p(q)l(\rho/q, t)d(\rho/q)qdq$, or, introducing $\sigma = \ln \rho$ and $\lambda(\sigma, t) = l(\exp \sigma, t)$,

$$\lambda(\sigma, t+1) = \int \lambda(\sigma - \ln q, t)p(q)dq. \tag{3}$$

In the case of 3D velocity fields, the same equation describes the evolution of fluid surfaces, with $\lambda(\sigma, t) = s(\exp \sigma, t)$. However, for a fixed velocity field, the stretching of fluid elements and surfaces are characterized by different functions $p(q)$.

The initial condition (2) implies that the dye gradients are initially delta-distributed. Therefore, in order to keep the equivalence between the stretching coefficients and dye gradients, the initial condition for Eq. (3) should be written as

$$\lambda(\sigma, 0) = \delta(\sigma). \tag{4}$$

This system can be solved by applying the Fourier transform $\lambda(f, t) = \int \lambda(\sigma, t) \exp(-if\sigma)d\sigma$. As a result we obtain

$$\lambda(f, t) = \Pi(f)^t, \Pi(f) = \int p(\exp q) \exp(q - ifq)dq. \tag{5}$$

On the long-time limit, the inverse Fourier transform can be taken via the saddle-point method:

$$\lambda \approx (\partial^2 F/t\partial f^2)^{-1/2} \exp\{F[f_0(h), h]t\}.$$

Here we have denoted $h = \sigma/t$, $F(f, h) = \ln[\int_{-\infty}^{+\infty} p(\exp q) \exp(q + fq)dq] - fh$, and $f_0(h)$ stands for the solution to the equation $\partial F/\partial f = 0$. This result has the same form as that of obtained for the PDF of largest Lyapunov exponents $p(h_m, t)$ via generalized Baker map model [3]. It should be stressed that although the functions $\lambda(h)$ and $p(h_m, t)$ have similar meaning, they are in fact distinct, even asymptotically at $t \to \infty$. This is caused by the contribution of fluid elements, almost perpendicular to the eigenvector of the largest Lyapunov exponent. Note that the function $F[f_0(h), h]$ is defined by the correlation properties of the velocity field. For a typical localized correlation function $p(k)$, function $F(f, h)$ grows linearly at $f \to \pm\infty$. Consequently, $f_0(h)$ is defined only for a finite range of the values of h. Outside of that region, $\lambda \equiv 0$. This is quite a natural result: for real non-Kraichnan flows with finite amplitude of velocity fluctuations, the stretching rate of a fluid element cannot be arbitrarily fast.

For Kraichnan flows and for real flows on the long-time limit, Eq. (5) can be further simplified. Indeed, on long-time limit, the function $\lambda(\sigma)$ has a smooth profile made up of long-wave-length Fourier components. Thus, the function $\Pi(f)$ can be expanded into power series, where only the two first terms are kept, $\Pi(f) \approx$

$-iuf - Df^2$. Here u and D are constants depending on the statistical properties of the velocity field:

$$u = \int p(k) \ln k \, dk, \quad D = \frac{1}{2} \int p(k)(\ln k)^2 dk. \tag{6}$$

Further, the discrete increments become relatively small and can be replaced by time derivative. Then, Eq. (3) can be rewritten as

$$\frac{\partial \lambda}{\partial t} + u \frac{\partial \lambda}{\partial \sigma} = D \frac{\partial^2 \lambda}{\partial \sigma^2}. \tag{7}$$

For the Kraichnan model, this is the exact result, which can be obtained via the Fourier transform of the expression $\lambda(\sigma, t) = \langle \delta[\sigma - \ln|\delta r(t)|] \rangle$. Here $\delta r(t) = \delta r(0) \exp(\int_0^t \nabla v \, dt')$, and the limit $t \to 0$ is to be studied. Then $\Pi(f) \equiv -iuf - Df^2$, with

$$u\delta(t - t') = \frac{d-1}{4} \langle [v_{r\tau}(t) + v_{\tau r}(t)][v_{r\tau}(t') + v_{\tau r}(t')] \rangle,$$
$$D\delta(t - t') = \frac{1}{2} \langle v_{rr}(t) v_{rr}(t') \rangle, \tag{8}$$

where indices r and τ denote the components of the tensor ∇v, and $d = 2, 3$ is the dimensionality of the space. For 3-dimensional stretching of fluid surfaces, $\lambda(\sigma, t) = \langle \delta[\sigma - \ln|\delta S(t)|] \rangle$, where $\delta S(t)^2 = \delta r_1(t)^2 \delta r_2(t)^2 - [\delta r_1(t) \cdot \delta r_2(t)]^2$, and $\delta r_{1,2}(t) = \delta r_{1,2}(0) \exp(\int_0^t \nabla v \, dt')$. Unlike in the case of finite correlation time flows, the values of u and D for 3-dimensional stretching of fluid surfaces are equal to the respective values for fluid elements. This is caused by two circumstances: first, the relevant infinitesimal increments of fluid elements satisfy the incompressibility condition; second, only the symmetric components of the stress tensor are involved. Typically, the stretching of fluid elements is defined by the largest Lyapunov exponent λ_+ (except for small fraction of fluid elements, almost perpendicular to the eigenvector). The stretching of fluid surfaces is governed by the sum $\lambda_+ + \lambda_0$, where λ_0 is the intermediate-valued Lyapunov exponent. Thus, the statistical equality of the two stretching coefficients means that the average value of λ_0 is zero. Finally we note that in the case of the Kraichnan model, the time is measured in arbitrary units, because the correlation time $\tau = 0$.

3   Multifractality of the passive scalar dissipation field

The passive scalar dissipation field, created by turbulent jet has been found to exhibit a multifractal structure [1]. This experimental finding has been addressed in several theoretical studies [2-4]. The analytic results confirm the presence of a multifractal structure. The stretching-coefficient-based approach has been used to calculate the multifractal spectrum $f(\alpha)$ for two-dimensional velocity field [9]. Here we extend this approach to 3-dimensional geometry.

Throughout this section, $\lambda(\sigma, t)$ will be treated as the PDF of dye gradients, i.e. $\sigma = \ln|\nabla \phi|$. We consider the initial conditions (2), which is equivalent to (4). Bearing this in mind, Eq. (7) can be immediately solved,

$$\lambda = (\pi D t)^{-1/2} \exp\left[-\frac{(\sigma - ut)^2}{Dt} \right]. \tag{9}$$

This expression allows us to derive the multifractal spectrum. It is convenient to make use of the pattern formed by fluid curves (or fluid surfaces — for 3-dimensional geometry), which were originally straight lines (plane surfaces), separated by unit length, and perpendicular to the gradient of the dye concentration. Note that the idea of studying a fluid line evolution has been used to calculate the Kolmogorov entropy in 2-dimensional quasi-stationary flow [14].

Experimentally, the multifractality has been observed on 1- and 2-dimensional cross-sections of the 3-dimensional dissipation field [1]. Here we shall study 1 and 2-dimensional cross-sections of 2- and 3-dimensional dissipation fields. We start with 1-dimensional cross-sections, particularly we consider the dependence of the local value of σ on the coordinate ξ along the cross-section. First we note that the characteristic fluctuation amplitude of the dye concentration is 1. Indeed, when the fluid lines (or fluid surfaces) evolve, they will be folded; typically, the density variations of the order of unity are embraced between two approaching each other pieces of the curve. Thus, on the cross-section, the characteristic scale of dye density variations is $\delta \approx 1/|\nabla\psi| \approx \exp(-\sigma)$. The small-scale variations of the function $\sigma(\xi)$ are described by the same scale. However, the function $\sigma(\xi)$ exhibits long-range correlations, as well, because two close each other pieces of a fluid curve are stretched in a similar way. It can be argued that in rescaled coordinates $\zeta = \int^\xi \xi' \exp[\sigma(\xi')]d\xi'$, function $\sigma(\zeta)$ is a random Brownian function. Indeed, the distance $\Delta\zeta$ between two fluid particles gives us the estimate, how long (i.e. how many durations of the correlation time) has been that time-period, when these points evolved in an uncorrelated manner. During each correlation time, a fluid element is randomly stressed or stretched leading to the change $\Delta\sigma \approx \pm1$. Therefore, $\Delta\zeta$ gives us the estimate, how many times are the fluid elements independently stressed or stretched.

Further, the multifractal structure of 1-dimensional cross-sections of 2- and 3-dimensional dissipation fields can be easily analyzed. The overall scalar dissipation in a region $[\xi, \xi+r]$, $r \lesssim 1$ can be estimated as $W_r(\xi) = \int_\xi^{\xi+r} \kappa|\nabla\psi|^2 d\xi \approx \kappa k_r^2 \delta_0 = \kappa k_r$, where k_r is the maximal value of $|\nabla\psi|$ over the given region, and $\delta_0 = 1/k_r$. In order to determine the multifractal spectrum $f(\alpha)$, we need to calculate the probability

$$p(r,\alpha) \propto r^{1-f(\alpha)} \qquad (10)$$

that the normalized dissipation $w_r(\xi) = W_r(\xi)/W_1(\xi)$ scales as α-th power of r, i.e. $w_r \in [r^\alpha, 2r^\alpha]$. Substituting the estimate $W_r(\xi) \approx \kappa k_r$, we obtain $W_r(\xi)/W_1(\xi) \approx k_r/k_1$. Then, the probability (10) can be calculated as

$$p(r,\alpha) = \begin{cases} L(\rho_0)r, & L(\rho_0)r \ll 1 \\ \exp[-L(\rho_0)r], & L(\rho_0)r \gg 1 \end{cases}, \rho_0 = k_{01}r^\alpha; \qquad (11)$$

where $L(\rho_0) = \int_{\rho_0}^\infty l(\rho)d\rho$ is the overall length [for 3-dimensional velocity field, this is the overall surface $L(\rho_0) = \int_{\rho_0}^\infty s(\rho)d\rho$] per unit area of those parts of the fluid curves (fluid surfaces), which are stretched more than a prefixed factor ρ_0. Indeed, $L(\rho_0)$ is the estimate for the number, how many times a cross-section of unit length is intersected by the fluid curves (fluid surfaces in 3D geometry) of stretching factor

$\rho > \rho_0$. According to (9),

$$L = \frac{1}{2} \exp\left[\left(u + \frac{D}{4}\right)t\right]\left[1 - \mathrm{erf}\frac{\sigma_0 + (u + D/2)t}{\sqrt{Dt}}\right], \tag{12}$$

where $\sigma_0 = \ln\rho_0$. At large values of σ_0, the asymptotics of Eq. (12) is given by $L \approx \exp[-(\sigma_0 + ut)^2/Dt - \sigma_0][1 + 2(\sigma_0 + ut)/Dt]^{-1}$; substituting $\sigma_0 = -\alpha|\ln r| + \ln k_{01}$ we obtain

$$p(r, \alpha) \approx r^{1-\alpha(\sqrt{1+4u/D} - \alpha|\ln r|/Dt)}. \tag{13}$$

Here we have also substituted the value of k_1, which has been calculated by noting that $p(1, \alpha) \approx 1$.

In its strict sense, multifractality assumes that $p(r, \alpha)$ is a power law of r. According to Eq. (13), this is valid only for small values of α, $\alpha \ll Dt\sqrt{1 + 4u/D}/|\ln(r_0)|$, where r_0 is the smallest considered space scale. In that case, expressions (10) and (13) yield

$$f(\alpha) = \alpha\sqrt{1 + 4u/D}. \tag{14}$$

It should be stressed that this expression assumes $Lr \ll 1$, and hence $f(\alpha) \lesssim 1$. On the other hand, slight deviations from multifractality in its strict sense may remain unnoticed when performing numerical schemes of obtaining multifractal spectra. Bearing this in mind, it makes sense to calculate the "effective" multifractal spectrum $f(\alpha)$, which might be obtained in experiments:

$$\langle f(\alpha) \rangle = \ln[p(r_0, \alpha)/r_0]/|\ln r_0|. \tag{15}$$
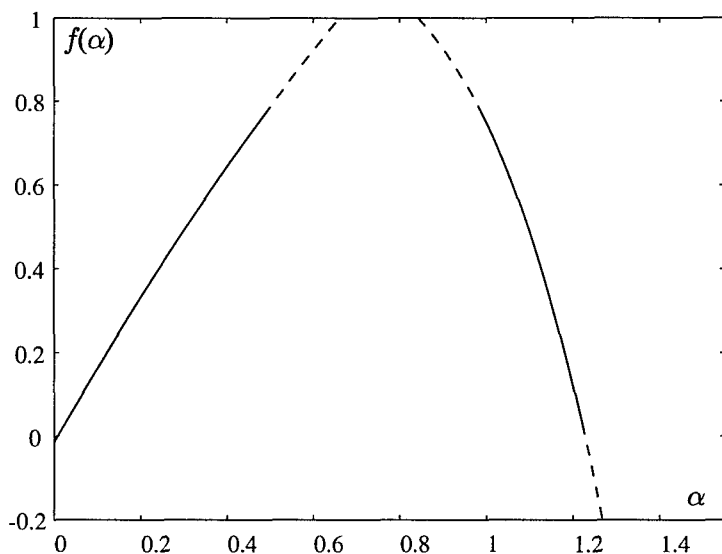

Figure 1. "Effective" multifractal spectrum $\langle f(\alpha) \rangle$, defined by expressions (11), (12) and (15); $u \approx 0.2$, $D \approx 0.35$, $r_0 \approx 0.01$, $t \approx 30$.

Further, we consider 2-dimensional cross-sections of 3-dimensional dissipation field. The cross-sections of the fluid surfaces are curves; in what follows, they will be referred to as "stripes". The variations of the dye gradient are highly anisotropic: the characteristic scale across the "stripes" is previously given by $\delta \approx \exp(-\sigma)$ and is typically defined by the smallest Lyapunov exponent h_-. Meanwhile, the characteristic scale along the "stripes" is much longer and is either defined by the intermediate-valued Lyapunov exponent h_0 (if $h_0 < 0$), or is of the order of unity (if $h_0 > 0$). The easiest way to obtain the $f(\alpha)$-curve is to consider 1-dimensional cross-sections of the 2-dimensional field. The fractal dimension of the intersection of two fractal objects is given by $D_\cap = D_1 + D_2 - D$, where D_1 and D_2 are the dimensionalities of the objects, and D is the topological dimension of the embedding space [15]. Before applying this relationship, it should be pointed out that on 1- and 2-dimensional cross-sections, the same physical points correspond to different values of α. Indeed, on 1-dimensional cross-sections, the normalized dissipation was estimated as k_r/k_1. For 2-dimensional geometry, the striped patterns lead to the estimate $w_r \approx rk_r/k_1 \approx r^\alpha$. By comparing 1- and 2-dimensional expressions for k_r/k_1 (expressed via r and α), one can see that 1-dimensional α corresponds to 2-dimensional $\alpha - 1$ [this correspondence is based on an implicit assumption that the intermediate-valued Lyapunov exponent is not very small, $h_0 t > -\ln(r_0)$]. Therefore, $f(\alpha)$ curves are related by the equality $f_2(\alpha) = 1 + f_1(\alpha - 1)$. Then, expression (14) leads to $f_2(\alpha) = 1 + (\alpha - 1)\sqrt{1 + 4u/D}$.

Finally we note that the curve defined by expressions (11), (12) and (15) is quite similar to the experimental curves, which have been obtained for the passive scalar dissipation in turbulent jet [1]. The leftmost part of the curve is linear, $f \propto \alpha$; this is the only part of the curve corresponding to a strict multifractality. Further, there is a rapid [according to Eq. (11) exponential] fall-off at the large values of α. Exact shape is not reproduced, because expression (11) has been obtained for two asymptotic limits; at the intermediate values of α, the assessment is very rough. Reasonable resemblance between the experimental [1] and theoretical curves is achieved for the following numerical values: $u \approx 0.2$, $D \approx 0.35$, $r_0 \approx 0.01$, $t \approx 30$ (see figure).

4   The measure of "harmfulness".

The convective diffusion is a phenomenon, which will appear in vast number of situations. One of the most important aspects is the mixing of harmful substances in natural environment. These substances may have been delivered as a result of an accident, or as technological wastes; the convecting medium may be oceanic or atmospheric motion. In most cases, the overall amount of the substance is relatively small, so that after complete mixing, there is no danger to the environment. Meanwhile, at the moderate (*under-mixing*) timescale, there are clumps of the substance; biological objects may become damaged, when hit by these clumps. It is not obvious, how to measure the harmfulness of this scalar field. The measures based on the moments of the scalar density are not suited. Indeed, at the under-mixing timescale, the seed diffusivity can be neglected, and one can use the reduced description, where the local state of the medium is given by one bit of information:

the substance is present ('1') or absent ('0'). The local average of the scalar concentration is also useless. This becomes evident, when we consider the evolution of a striped pattern. Due to the incompressibility of the velocity field, the average concentration of the dye will be quasi-homogeneous: in those places, where the stretching factor is high, the stripe becomes very thin; at the same time, it will be approached by another piece of stripe, so that the amount of dye per unit volume will remain unchanged.

Intuitively it is clear that the quantity which does matter, is the size of the blob of dye. Thus, we define new scalar field $\chi(r, t)$ as the radius of the sphere, which is completely immersed into the dye, and the center of which is at the given point. This field — or an n-th power of it $\chi^n(r, t)$ — can be taken as the measure for the harmfulness of the admixture. While $\chi^0(r, t)$ ("zero order harmfulness") is essentially equivalent to the dye concentration itself and is distributed quasi-homogeneously, all the other moments reveal multifractal structure.

In order to study the multifractality of the "harmfulness", we follow the approach of Section 3. Thus, we consider 1-dimensional cross-sections of the scalar field. For the sake of simplicity, let us study only 2-dimensional velocity field (3-dimensional geometry can be handled in the same way, as in the case of dissipation field). Suppose that initially, the stripes of dye of width ϵ were straight lines separated by unit distance. Then, the total n-th order "harmfulness" over a region of size r is estimated as $H_n = \int_\xi^{\xi+r} \chi^n d\xi \approx \epsilon^n \tilde{k}_r^{-n-1}$, where $\tilde{k}_r$ is the smallest value of $|\nabla\psi|$. Note that this estimate is valid for $n > 0$ (actually, for n not too close to 1), because we have taken into account only the contribution of the largest blob of dye in the given region. Strictly speaking, the sum should have been taken over all the blobs. However, just like in the case of dissipation fields, for $n > 0$, the sum is dominated by the largest term. Further, the multifractal spectrum of dissipation fields was derived using the overall length per unit area L of those parts of the fluid curve, which were stretched more than a prefixed factor. Here we have a symmetric situation: the main contribution to the "harmfulness" is made by those parts of the stripes, which have small stretching factors. Therefore, the probability $p(r, \alpha)$ can be assessed by the same formula (11) as in the case of dissipation fields; the only modification is that L should be substituted by $\tilde{L}$ — the length of those parts of the stripes, which are stretched less than $\tilde{k}_1 r^{-\alpha/(n+1)}$ times. Following the procedure of obtaining Eq. (14), we result in

$$f_n(\alpha) = \frac{\alpha}{n+1}\sqrt{1 + 4u/D}. \qquad (16)$$

Note that while the multifractality described by Eq. (14) is caused by the "right-hand tail" of the Gaussian distribution function (9) — i.e. by the fall-off at $\sigma \to +\infty$ — the multifractality of the "harmfulness" is due to the left-hand Gaussian tail. Therefore, it is not surprising that Eqns. (14) and (16) are very similar to each other.

5   Conclusions

We have introduced the concept of "harmfulness", which addresses the problem of environmental damage caused by technological wastes. We have shown that in

the case of passive scalar turbulence, both the multifractality of the dissipatj field and "harmfulness" are caused by uneven stretching of fluid elements. For multifractality of the dissipation field, this is a known result [2,4]. The advantag the approach based on Eq. (7) is its simplicity, which made it possible to calcu the $f(\alpha)$-curves. Eq. (7) can be also used to study the other aspects of pas scalar turbulence. Thus, it has been modified to take into account the non-molecular diffusivity [9]. In such a way, it was possible to address the problem stationary $1/k$ power spectrum, and exponential decay of the dye fluctuatior the case of initially seeded dye. However, it seems that the approach canno extended to study the PDF of dye gradients in the presence of non-zero diffusiv the dye gradients are affected by the coherence between the existing patterns the patterns formed by diffusing dye.

Acknowledgments

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